J. Fluid Mech. (1967), *vol.* 30, *part* 1, *pp.* 33–49
Printed in Great Britain

Convection in horizontal layers with internal heat generation. Theory

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(Received 24 January 1967)

A theoretical study has been made of an experiment by Tritton & Zarraga (1967) in which convective motions were generated in a horizontal layer of water (cooled from above) by the application of uniform heating. The marginal stability problem for such a layer is solved, and a critical Rayleigh number of 2772 is obtained, at which patterns of wave-number 2·63 times the reciprocal depth of the layer are marginally stable.

The remainder of the paper is devoted to the finite amplitude convection which ensues when the Rayleigh number, R, exceeds 2772. The theory is approximate, the basic simplification being that, to an adequate approximation, Fourier decompositions of the convective motions in the horizontal (x, y) directions can be represented by their dominant (planform) terms alone. A discussion is given of this hypothesis, with illustrations drawn from the (better studied) Bénard situation of convection in a layer heated below, cooled from above, and containing no heat sources. The hypothesis is then used to obtain 'mean-field equations' for the convection. These admit solutions of at least three distinct forms: rolls, hexagons with upward flow at their centres, and hexagons with downward flow at their centres. Using the hypothesis again, the stability of these three solutions is examined. It is shown that, for all R, a (neutrally) stable form of convection exists in the form of rolls. The wave-number of this pattern increases gradually with R. This solution is, in all respects, independent of Prandtl number. It is found, numerically, that the hexagons with upward motions in their centres are unstable, but that the hexagons with downward motions at their centres are completely stable, provided R exceeds a critical value (which depends on Prandtl number, P, and which for water is about $3R_c$), and provided the wave-number of the pattern lies within certain limits dependent on R and P.

1. Introduction

In the first paper with this title, Tritton & Zarraga (1967) reported the results of qualitative experiments on convection in a horizontal layer of fluid cooled from above, thermally insulated from below, and heated nearly uniformly by electrolytic currents from within. Two striking results emerged. First, the cell structure was, for moderate Rayleigh numbers, R, predominantly hexagonal with motion *downwards* at the centre of each cell. Secondly, the horizontal scale of the convection pattern grew larger as R was increased above its critical value,

R_c. This paper represents a preliminary theoretical attempt to answer these experimental challenges, and, at the same time, to throw more light on the advantages and limitations of an approximate theory of finite amplitude convection (Roberts 1966; this paper will henceforward be referred to as Paper I).

The steady finite amplitude convection that occurs for $R > R_c$ will be studied by a variational method due to Glansdorff & Prigogine (1964). In selecting trial functions for this method, we suppose that the structure of the convection is, in its horizontal Fourier (x, y) structure, identical to that which occurs at marginal convection. In place of the partial differential equations in x, y and z, we obtain ordinary differential equations in z, the vertical co-ordinate. These, in the case of convection in rolls, are identical with the so-called 'mean-field equations', and we shall so term them here. The stability of finite amplitude solutions may also be examined by the variational methods of Prigogine & Glansdorff (1965). The choice of a similar simple trial function for the perturbation again results in ordinary differential equations for the normal modes.

Quite apart from the fact that these mean-field equations (and those governing the stability of their solutions) are simple to interpret, they can be solved with comparative ease.† It is important, therefore, to gain experience of their reliability and their limitations. One may feel that, although they provide a reasonable way of dealing with steady convection, they may be too gross to master the subtleties raised by questions pertaining to the preferred mode. It is, however, worth recording that, when applied to the classical Bénard situation,‡ the method successfully predicted that, if the changes of the physical constants (such as viscosity) with temperature are negligible, the preferred mode is, at least for small R, in rolls and not hexagons (see Paper I, p. 147; see also Rossby 1966 and Somerscales & Dropkin 1966). It is also clearly of some interest to know whether we can use the method to account for Tritton & Zarraga's results. The answer appears to be that it cannot do so in all respects. It agrees with the experiments in the sense that hexagons with upward motions at their centres are ruled out, and in the sense that hexagons with downward motions at their centres are stable, provided R is sufficiently large (> 8750, for water). It conflicts with the experiments in the sense that it does not account for the decrease in wave-number, a, of the pattern that Tritton & Zarraga observe when R is increased; indeed, on the theory presented here, hexagonal patterns for their values of a should be unstable (though not violently).

Before concluding that the present approximation is inadequate because it fails to predict the cell sizes correctly, we should consider an alternative possibility: the basic model (cf. §2 below) may itself be an inadequate replica of the experimental situation. Since the electrical resistivity, s, of the working fluid

† For example, in §4 below, we report results derived, to an accuracy of 1 part in 10^3, for the solution of the full partial differential equations for Bénard convection in rolls. This absorbed some 20 h of KDF 9 computer time. The corresponding solution of the mean-field equations, to an accuracy of 1 in 10^6, took only 20 min.

‡ That is, convection in a layer heated from below, cooled from above, in the absence of internal heat sources. In the following work we will, for brevity, always refer to this as 'classical Bénard convection' when we wish to distinguish it from the Tritton–Zarraga situation.

depends on temperature, T, the uniformity of heating, presupposed in the theory, was not realised in the experiments. It is readily shown that, if this effect is included, the critical Rayleigh number is changed by the order of

$$\epsilon = (\gamma d^2/\kappa)(ds/s\,dT),$$

where d is the depth of the layer, κ is the thermal diffusivity, and γ is the mean rate at which the temperature of the layer would rise in the absence of heat conduction. Moreover, a preferred direction is introduced, that of the electric current heating the fluid. There is, for example, a difference of order ϵ in the Rayleigh number for rolls aligned with the current and that for rolls perpendicular to it. The value of ϵ in the experiments appears to be about $10^{-5}R$, and we feel confident that its effect on the structure of the finite amplitude solutions would be too small to be felt. For experimental corroboration, we observe that there was no evidence of a preferred horizontal direction in Tritton & Zarraga's observations. Further, the question of the sense of motion within hexagonal cells appears to be associated with the gross asymmetry of the layer, and this is present even if $\epsilon = 0$. It seems unlikely, then, that a small finite value of ϵ would affect the conclusion. On the other hand, the question of the preferred cell size is far more subtle, and it is not inconceivable that the ϵ effect would make itself felt. It would, however, be remarkable if it could resolve the discrepancy noted above. Other differences between theory and experiment arise from the electromagnetic forces absent in the former, but present in the latter. As Tritton & Zarraga show, however, these are so minute that they cannot credibly affect even questions of cell size.

2. Basic equations: linear stability

Our model consists of a uniform horizontal layer, $0 < z < d$, unbounded in the x and y directions, and containing a fluid of density ρ, kinematic viscosity ν, thermal diffusivity κ, and coefficient of volume expansion α. Its lower surface ($z = 0$) is in contact with a rigid, thermally insulating, horizontal slab. Its upper surface is in contact with a rigid, conducting, plane which is maintained at a constant temperature (which defines the zero point of our scale). Uniform sources produce heat in the layer which, in the absence of conduction, would cause the temperature of every fluid element to rise at a constant rate, γ. (Note that $\gamma = H/\rho c_p$ in Tritton & Zarraga's notation.)

We use dimensionless variables. We will measure time, t, in units of d^2/ν, length in units of d, but (nevertheless) velocity, $\mathbf{u}$, in units of κ/d (and not ν/d). We will measure temperature, T, from its zero on $z = 0$, in units of $\gamma d^2/\kappa$. We will denote by P and R the Prandtl and Rayleigh numbers

$$P = \frac{\nu}{\kappa}, \quad R = \frac{g\alpha\gamma d^5}{\nu\kappa^2}, \tag{1}$$

where g is the acceleration due to gravity.

The basic equations governing the study are now seen to be, in the Boussinesq approximation,

$$\operatorname{div}\mathbf{u} = 0, \tag{2}$$

$$\frac{\partial\mathbf{u}}{\partial t}+\frac{1}{P}\mathbf{u}.\operatorname{grad}\mathbf{u} = -\operatorname{grad}\varpi+\nabla^2\mathbf{u}+RT\mathbf{1}_z, \tag{3}$$

$$P\frac{\partial T}{\partial t}+\mathbf{u}.\operatorname{grad}T = \nabla^2 T+1, \tag{4}$$

where ϖ is a dimensionless variable representing pressure, and $\mathbf{1}_z$ is a unit vector in the direction of z-increasing, i.e. upwards. The boundary conditions are

$$DT = 0, \quad \text{on} \quad z = 0; \quad T = 0, \quad \text{on} \quad z = 1; \tag{5}$$

$$\mathbf{u} = 0, \quad \text{on} \quad z = 0 \quad \text{and} \quad z = 1; \tag{6}$$

where $D = \partial/\partial z$.

Steady solutions of (2) to (6) exist, of the form

$$\mathbf{u} = 0, \quad T = T^{(0)} \equiv \tfrac{1}{2}(1-z^2), \quad \varpi = \varpi^{(0)} \equiv R(k+\tfrac{1}{2}z-\tfrac{1}{6}z^3), \tag{7}$$

where k is an arbitrary constant. These are called *conduction solutions*. If R is sufficiently great, steady solutions of the form

$$\mathbf{u} \neq 0, \quad T = T_0(z)+\theta(x,y,z), \quad \varpi = \varpi_0(z)+\Pi(x,y,z) \tag{8}$$

also exist in which the averages, $\langle\mathbf{u}\rangle$, $\langle\theta\rangle$ and $\langle\Pi\rangle$, of $\mathbf{u}$, θ and Π over the horizontal plane vanish. These are the *convection solutions*; the functions $\mathbf{u}$, θ and Π represent the tessalated motions of convection, and T_0 and ϖ_0 represent the corresponding conduction terms $T^{(0)}$ and $\varpi^{(0)}$ as modified by these motions.

If $\mathbf{u}$, θ and Π are infinitesimal, it may be shown (see, for example, Chandrasekhar 1961, chapter 2) that (8) is of the form

$$\mathbf{u} = \left[\frac{DW(z)}{a^2}\frac{\partial f(x,y)}{\partial x}, \frac{DW(z)}{a^2}\frac{\partial f(x,y)}{\partial y}, W(z)f(x,y)\right], \tag{9}$$

$$T = T_0(z)+F(z)f(x,y), \quad \varpi = \varpi_0(z)+\Pi(z)f(x,y), \tag{10}$$

where

$$T_0(z) = T^{(0)}(z), \quad \varpi_0(z) = \varpi^{(0)}(z). \tag{11}$$

Here $f(x,y)$ satisfies

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2} = -a^2 f, \quad \langle f^2\rangle = 1. \tag{12}$$

The wave-number, a, decides the horizontal scale of the convective motions; the type of solution to (12) selected decides their horizontal structure, or 'planform'. For example, for convection in rolls of width π/a having axes in the y-direction, we must take $f = \sqrt{2}\cos ax$; convection in hexagons is represented by a planform which is the sum of those for three sets of rolls at 60° to each other. It may be noted that, by (12), the velocity (9) automatically obeys (2).

By a process too familiar to bear repetition here, it may be shown from (3), (4), (9) and (11) that, irrespective of the solution to (12) selected, we have

$$(D^2-a^2)^2 W \equiv Ra^2 F, \tag{13}$$

$$(D^2-a^2)F = WDT_0, \tag{14}$$

where, by (5) and (6),

$$W = DW = DF = 0 \quad \text{on} \quad z = 0; \quad W = DW = F = 0 \quad \text{on} \quad z = 1. \tag{15}$$

Note that, by (7) and (11),

$$DT_0 = -z. \tag{16}$$

The solution to (13) to (15), for given a, poses an eigenvalue problem for R. The smallest eigenvalue, $R(a)$ (say), decides the onset of convection for that value of a. The minimum R_c of $R(a)$, which occurs at a_c (say), decides the critical Rayleigh

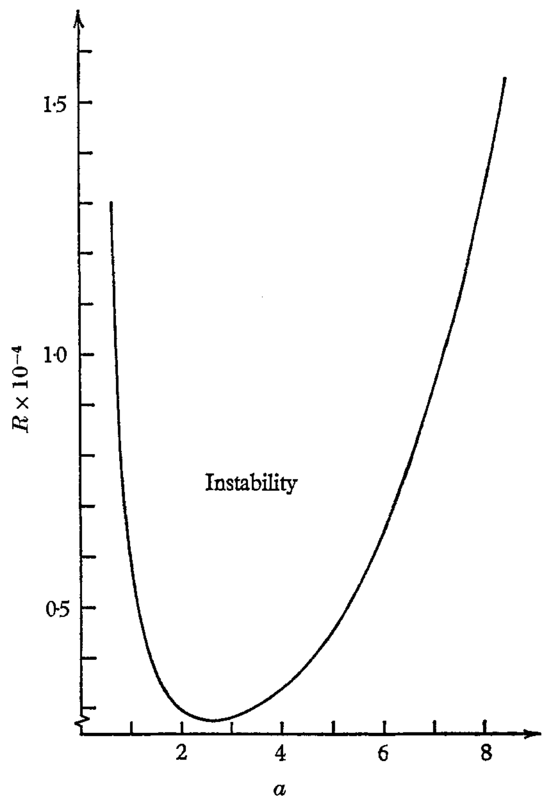

FIGURE 1. The neutral stability curve. For each value of the wave-number, a, the figure gives the smallest eigenvalue, R, of (13) to (16).

number and wave-number at which convection first occurs as R is increased gradually from zero. The problem of solving (13) and (14) for a general linear profile [which includes (16)] has been considered by Sparrow, Goldstein & Jonsson (1964), and by Debler (1966).† Unfortunately, however, they did not

† Debler showed that the adjoint of the Sparrow, Goldstein, Jonsson convection problem is the much-calculated problem of the onset of instability in Couette flow in the case of a narrow gap. This analogy suggests that one might regard the present problem as being approximated by an averaged temperature gradient, β, of $\gamma d/2\kappa$. This yields a critical classical Rayleigh number, $g\alpha\beta d^4/\nu\kappa$, of $\frac{1}{2}(2772) \approx 1400$, which is roughly the reduction below 1700 one might expect in view of the difference in boundary conditions. I am grateful to Professor J. T. Stuart for these observations.

include conditions (15) in their discussion. We have therefore had to perform the calculations afresh, obtaining the curve $R(a)$ shown in figure 1. It has a single minimum at

$$R_c = 2772{\cdot}28, \quad a_c = 2{\cdot}629,$$

corresponding, for the case of convection in rolls, to a horizontal semi-wavelength of $1{\cdot}195d$. As in the classical Bénard situation, $R = O(a^{-2})$ as $a \to 0$, and $R \sim a^4$ as $a \to \infty$. In fact, numerical calculation shows that $R \sim 4057{\cdot}73a^{-2}$, as $a \to 0$. Straightforward asymptotic analysis shows that, in the case $a \to \infty$, convection occurs predominantly in a layer of thickness $a^{-\frac{4}{3}}$ adjacent to the upper surface.

3. Integral properties: the shape assumption

We return to the finite-amplitude problem posed by (2) to (6). On making the substitution (8), it may be shown (cf. Paper 1, or Chandrasekhar 1961, appendix 1) that,

$$\int_0^1 \langle \theta u_z \rangle DT_0 \, dz = -\int_0^1 \langle (\nabla\theta)^2 \rangle \, dz, \tag{17}$$

$$R\int_0^1 \langle \theta u_z \rangle \, dz = \int_0^1 \left\langle \left(\frac{\partial u_i}{\partial x_j}\right)^2 \right\rangle dz, \tag{18}$$

and

$$DT_0 = -z + \langle \theta u_z \rangle, \tag{19}$$

implying that

$$T_0 = \tfrac{1}{2}(1-z^2) - \int_z^1 \langle \theta u_z \rangle \, dz. \tag{20}$$

By (17) and (19)

$$\int_0^1 \langle \theta u_z \rangle^2 dz = \int_0^1 \langle \theta u_z \rangle z \, dz - \int_0^1 \langle (\nabla\theta)^2 \rangle \, dz. \tag{21}$$

The shape assumption (Stuart 1958) is the supposition that, to an adequate approximation,

$$\theta = \frac{AR_c}{R}\theta_c, \quad \mathbf{u} = A\mathbf{u}_c, \tag{22}$$

where θ_c and $\mathbf{u}_c$ denote an arbitrarily normalized solution of the marginal solution of the perturbation equations of § 2. The equations (18) and (21) are satisfied by θ_c and $\mathbf{u}_c$, and the factor R_c/R in (22) ensures that θ and $\mathbf{u}$ obey (18) automatically. The remaining condition (21) then gives

$$A^2 = \Lambda(R - R_c), \tag{23}$$

where

$$\Lambda = \int_0^1 \langle \theta_c u_{zc} \rangle z \, dz \Big/ R_c \int_0^1 \langle \theta_c u_{zc} \rangle^2 dz. \tag{24}$$

In classical Bénard convection, the shape assumption has been used to estimate the Nusselt number N, which is defined to be the mean heat flux through the convecting layer divided by the heat flux through the same layer when immobilized. In the present situation, however, N is inappropriate since it is necessarily unity. The complementary quantity here appears to be a parameter, M, defined to be the mean temperature of the bottom surface of the convecting layer divided by its temperature when the same layer is immobilized:

$$M = 2T_0(0) = 1 - 2\int_0^1 \langle \theta u_z \rangle \, dz. \tag{25}$$

(In the classical Bénard situation, M is necessarily unity.) On the basis of the approximation (22) and (23), we see that

$$M = 1 - \frac{\Gamma}{R}(R - R_c), \tag{26}$$

where Γ is independent of the normalization of θ_c and $\mathbf{u}_c$ selected, and is given by

$$\Gamma = 2\int_0^1 \langle \theta_c u_{zc} \rangle dz \int_0^1 \langle \theta_c u_{zc} \rangle z\, dz \Big/ \int_0^1 \langle \theta_c u_{zc} \rangle^2 dz. \tag{27}$$

It is found that, for $a = a_c$,

$$\Gamma = 0{\cdot}5994. \tag{28}$$

For the remainder of this paper, we abandon assumption (22). We should note, however, that the theory to be presented obeys (19) and the very fundamental integral relations (17) and (18).

4. The mean-field approximation

We have already seen that, in marginal convection, (8) reduces to the form given in (9) to (11), which involves only one (fundamental) wave-number, a. Having selected a value of a and a corresponding solution of (12), it is easily shown that, in finite amplitude convection, this fundamental harmonic generates, via the $\mathbf{u}$.grad terms of (3) and (4), an infinite sequence of 'overtones' whose wave-numbers are simple multiples of a. For example, for convection in rolls, the sequence of wave-numbers is a, $2a$, $3a$, $4a$,...; for convection in hexagons it is a, $\sqrt{3}a$, $2a$, $\sqrt{7}a$, $3a$, $\sqrt{12}a$, $\sqrt{13}a$, $4a$... (Platzman 1965).† Since viscosity becomes increasingly dominant as we progress to the right along the sequence, the amplitude of the harmonics must decrease 'rapidly'. To illustrate this, we consider the case of convection in rolls in which $\mathbf{u}$ may be represented by a stream function, ψ, such that $\mathbf{u} = (-\partial\psi/\partial z, 0, \partial\psi/\partial x)$, and ψ and T may be expanded as

$$\psi = \sum_{s=1}^{\infty} \psi_s(z)\sin sax, \quad T = \sum_{s=0}^{\infty} T_s(z)\cos sax. \tag{29}$$

In table 1, the values of ψ_s and T_s are shown, not for the present situation, but for the (better-studied) case of classical Bénard convection. The units are those of the present paper, but the boundary conditions on T are: $T = 0$ on $z = 0$, and $T = -1$ on $z = 1$; also, the γ source of §2 is absent. Values are given for $0 \leqslant z \leqslant \frac{1}{2}$ only, since those for $\frac{1}{2} \leqslant z \leqslant 1$ follow directly from them by symmetry. The case

† In the case of rolls, each term of the velocity series is poloidal, i.e. it can be represented in the form (9) with its wave-number replacing a in (9) and (12). In the case of hexagons, however, the velocity series contains toroidal terms in addition, with wave-numbers $\sqrt{7}a$, $\sqrt{13}a$, $\sqrt{19}a$, $\sqrt{21}a$...[A toroidal velocity is expressible in the form $\mathbf{u} = (W\partial g/\partial y, -W\partial g/\partial x, 0)$ where $g(x, y)$ obeys (12) with the appropriate wave-number replacing a.] In hexagonal motions, therefore, the z-component of vorticity is non-zero (though weak). In the case of classical Bénard convection, Platzman showed that the dominant wave-number, a, must lie above the neutral stability curve. In the present case, however, this will not be so. There will exist close to the curve of figure 1, a range of R in which sub-critical finite amplitude hexagonal solutions exist (Busse 1967). (I am grateful to Dr F. Busse for pointing this out.) This paper has, however, been concerned mainly with experiments at highly super-critical R.

considered is $a = 3$, $P = 6{\cdot}7$ and $R = 20{,}000(\approx 12R_c)$. It will be seen that, as expected, ψ_s and T_s for any fixed z decrease rapidly as z increases. It is not known, at the present time, whether similar series for hexagonal motions, or for the situations considered in the present paper, would converge so rapidly, but there is no obvious reason why they should not.

$(-1)^{s+1}\psi_s(z)$

z \ s	1	2	3	4	5	6	7	8	Approximate 1
0·1	2·25	0·036	0·26	0·049	0·03	0·008	0·00	0·001	2·56
0·2	6·61	0·124	0·67	0·109	0·07	0·018	0·01	0·002	7·63
0·3	10·86	0·173	1·04	0·123	0·10	0·020	0·01	0·003	12·50
0·4	13·78	0·121	1·30	0·080	0·12	0·013	0·01	0·002	15·80
0·5	14·80	0	1·40	0	0·12	0	0·01	0	16·96

$(-1)^{s+1}T_s(z)$

z \ s	0	1	2	3	4	5	6	7	8	Approximate 0	Approximate 1
0·0	0	0	0	0	0	0	0	0	0	0	0
0·1	0·296	0·151	0·013	0·022	0·011	0·006	0·003	0·002	0·000	0·330	0·207
0·2	0·480	0·197	0·040	0·038	0·020	0·012	0·006	0·003	0·001	0·492	0·255
0·3	0·524	0·166	0·044	0·052	0·020	0·017	0·007	0·004	0·001	0·507	0·190
0·4	0·519	0·138	0·024	0·063	0·012	0·020	0·005	0·004	0·001	0·402	0·148
0·5	0·500	0·129	0	0·067	0	0·020	0	0·004	0	0·500	0·139

TABLE 1. x-Fourier coefficients of stream function and temperature as functions of z (classical Bénard convection)

Our basic approximation (cf. Paper I) consists in truncating Fourier expansions such as (29) after their first terms, i.e. we will suppose that, to an adequate accuracy, we may represent the solution we seek by (9) and (10). We do *not*, however, suppose that (11) holds, i.e. we allow for the modifications in the mean temperature profile created by convection. We term the resulting *ordinary* differential equations for W, F and T_0 'the mean field equations', since this is what they are in the case $C = 0$.

At first sight, the approximation just introduced appears to be inconsistent. Imagining a *complete* Fourier decomposition of the motions in x, y *and* z, one might ask why one should truncate the (x, y) series of coefficients after their first terms while keeping (essentially) all the corresponding z-Fourier coefficients? For, at large R, it appears (Pillow 1949) that the boundary-layer thickness at the side walls of a convection cell is essentially the same as that at its upper and lower faces; the rate of convergence of the (x, y) coefficients should, then, be no greater than that of the z harmonics. On the other hand, it must be agreed that, at $R = R_c$, our approximation exactly reproduces the perturbation equations†; the

† As $R \to R_c + 0$, the mean field equations give qualitatively correct results. For example, Busse (1967), examining the effects of deviations from the Boussinesq equations on classical Bénard convection, concluded that the error in the heat flux was only 4% for hexagons and 1% for rolls (rigid boundaries).

double Fourier series for u_z and θ in the marginal case each possess one term in the (x, y) 'direction', but an infinite number (cf. Jeffreys & Jeffreys 1946, § 14.062) in the z 'direction'. It may, therefore, be not unreasonable to suppose that, at 'intermediate' values of R, the double Fourier series will converge far more rapidly in the (x, y) direction than in the z direction. We may offer, in confirmation, the following example taken from the solution to the full non-linear equations of classical Bénard convection in rolls. Writing

$\psi_{s,t}$

$t \backslash s$	1	2	3	4	5	6	7	8
1	13·75	0	1·32	0	0·12	0	0·01	0
2	0	−0·152	0	−0·120	0	−0·020	0	−0·001
3	−1·65	0	−0·12	0	−0·01	0	0·00	0
4	0	0·045	0	0·015	0	0·003	0	0·000
5	−0·50	0	−0·04	0	0·00	0	0·00	0
6	0	0·010	0	0·006	0	0·001	0	0·000
7	−0·20	0	−0·02	0	0·00	0	0·00	0
8	0	0·002	0	0·003	0	0·000	0	0·000
9	−0·09	0	−0·01	0	0·00	0	0·00	0
10	0	0·001	0	0·002	0	0·000	0	0·000
11	−0·05	0	−0·01	0	0·00	0	0·00	0

$T_{s,t}$

$t \backslash s$	0	1	2	3	4	5	6	7
0	−0·500	−0·145	0	−0·042	0	−0·013	0	−0·003
1	0·168	0	0·031	0	0·017	0	0·005	0
2	0	0·012	0	0·027	0	0·009	0	0·002
3	0·151	0	−0·026	0	−0·011	0	−0·004	0
4	0	0·050	0	0·004	0	0·002	0	0·001
5	0·072	0	−0·006	0	−0·003	0	−0·001	0
6	0	0·031	0	0·003	0	0·001	0	0·000
7	0·034	0	0·001	0	−0·001	0	0·000	0
8	0	0·016	0	0·002	0	0·000	0	0·000
9	0·018	0	0·001	0	0·000	0	0·000	0
10	0	0·008	0	0·001	0	0·000	0	0·000
11	0·012	0	0·000	0	0·000	0	0·000	0
12	0	0·006	0	0·001	0	0·000	0	0·000

TABLE 2. (x, z)-Fourier coefficients of stream function and temperature (classical Bénard convection)

$$\psi = \sum_{s,t=1}^{\infty} \psi_{s,t} \sin sax \sin t\pi x, \quad T = \sum_{s,t=0}^{\infty} T_{s,t} \cos sax \cos t\pi z, \tag{30}$$

we find, in the case $a = 3$, $P = 6{\cdot}7$ and $R = 20{,}000$, the values of $\psi_{s,t}$ and $T_{s,t}$ shown in table 2. It will be observed that the ratio of successive terms in $2s$ or $2s+1$ for any fixed t is ultimately of the order of $0{\cdot}1$; the ratio of terms in $2t$ or $2t+1$ for fixed s is ultimately, however, of order $0{\cdot}5$. As further corroboration we show, in the final column of table 1, the values of ψ_1, T_0 and T_1 according to the mean-field theory. These should be compared with the corresponding (exact) results for ψ_1, T_0 and T_1 in table 1. (The approximate value obtained for N was 3·51; this should be compared with the exact value of 3·13.)

We now return to the internal heat generation problem. The derivation of the mean field equations follows too closely that given in Paper I to be repeated here. They are found to be [cf. Paper I, equations (49) to (54)]

$$(D^2-a^2)^2W = Ra^2F+(C/P)[WD(D^2-a^2)W+2DW(D^2-a^2)W], \tag{31}$$

$$(D^2-a^2)F = WDT_0+C[FDW+2WDF], \tag{32}$$

$$DT_0 = -z+FW, \tag{33}$$

where

$$W = DW = DF = 0 \quad \text{on} \quad z=0; \quad W = DW = F = 0 \quad \text{on} \quad z=1; \tag{34}$$

and C denotes the coupling constant $\langle f^3\rangle/2$, which takes the value $1/\sqrt{6}$ for hexagonal convection and is zero for convection in rolls. We may note that, when $C = 0$, (31), (32) and (35) are identical with (13) to (15), though, of course, (33) differs from (16). From (33) we have [cf. (20) and (25) above]

$$T_0 = \tfrac{1}{2}(1-z^2)-\int_z^1 FW\,dz, \tag{35}$$

$$M = 1-2\int_0^1 FW\,dz. \tag{36}$$

It may readily be verified that solutions to (31) to (36) automatically obey the integral requirements of §3 for all C. For $C = 0$, they are the usual mean-field equations (cf. Herring 1963).

We will consider the solutions to (31) to (36) in three† distinct cases; viz. convection in rolls, convection in hexagons with motions rising in the centre of each cell, and convection in hexagons with motions descending at each centre. We will refer to these as rolls, up-hexagons and down-hexagons, respectively. In figures 2 and 3, the forms of $W(Ra^2)^{-\frac{1}{2}}$, $F(Ra^2)$ and T_0 are shown for one value of R for the roll and both hexagons, and in the two limiting cases $R \to R_c$, and $R \to \infty$ (for rolls).‡ It will be noted, in each case, that the solutions are asymmetric with respect to the mid-plane of the layer.

The asymptotic case $R \to \infty$ for convection in rolls ($C = 0$) may be solved easily by the analysis instituted by Roberts (Paper I) and completed by Stewartson (appendix to Paper I). The boundary layer at $z = 1$ is essentially of the same type as occurred at both surfaces in the classical Bénard situation; its thickness is of order $(Ra^2 \log Ra^2)^{-\frac{1}{4}}$. The boundary layer at $z = 0$ is of different structure; its thickness is of order $(Ra^2)^{-\frac{1}{5}}$. In evaluating

$$M = \frac{2}{Ra^2}\int_0^1 [W(D^2-a^2)^2W-Ra^2z]\,dz, \tag{37}$$

† In the corresponding Bénard case, these reduce to two, since there is no theoretical distinction between up and down hexagons: see discussion below equation (55) of Paper I.

‡ These results and those reported below were obtained on the KDF 9 computer of the University of Newcastle upon Tyne using a Chebyshev collocation method (cf. Wright 1964).

[cf. (31) and (36)], we see that the dominant contribution arises from the $z = 1$ boundary layer, which yields

$$M \sim 2{\cdot}221(Ra^2 \log Ra^2)^{-\frac{1}{4}}. \tag{38}$$

As Roberts has shown (Paper I), these results have to be modified if $a = O(R^{\frac{1}{4}})$.

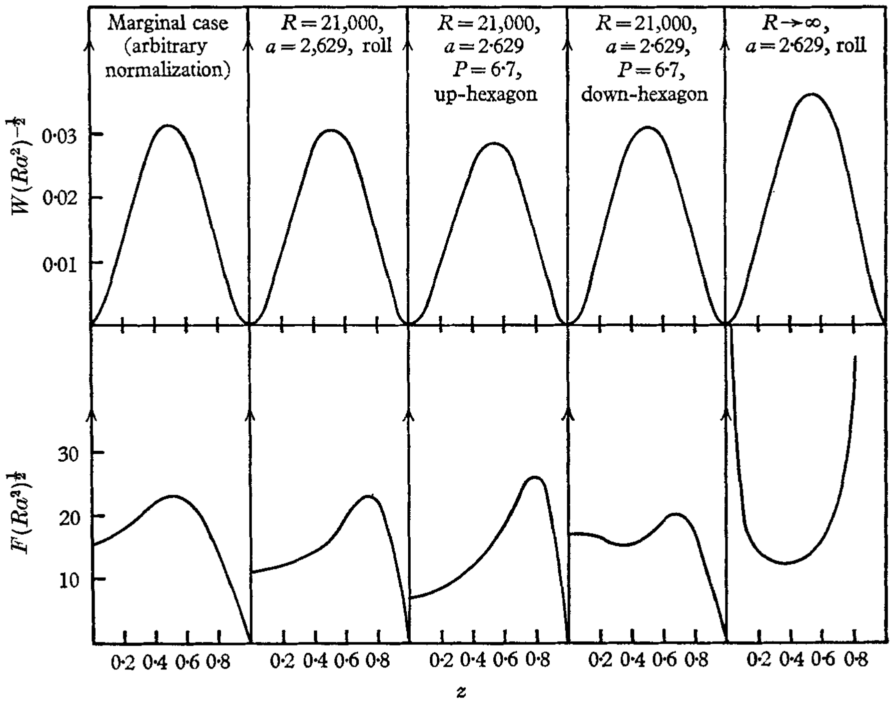

FIGURE 2. Illustrative solutions of the mean field equations. The figure shows W and F for (left to right): the marginal case $a = a_c$, $R = R_c + 0$; a roll in the case $R = 21{,}000$, $a = a_c$; an up-hexagon in the same case ($P = 6{\cdot}7$); a down-hexagon in the same case; a roll in the limit $R \to \infty$.

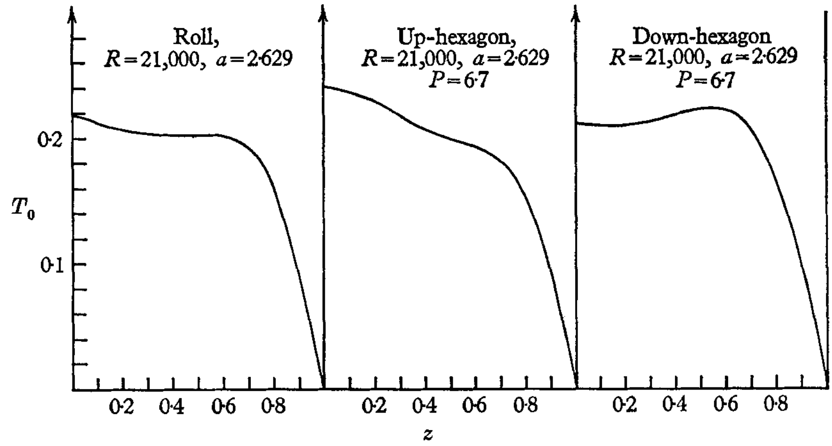

FIGURE 3. Illustrative solutions of the mean field equations. The figure shows T_0 for (left to right): a roll in the case $R = 21{,}000$, $a = a_c$; an up-hexagon in the same case ($P = 6{\cdot}7$); a down-hexagon in the same case.

5. The preferred modes

In this section we report on the results obtained by applying a theory of the preferred mode developed in Paper I to the present problem. We will not, however, repeat the formalism in detail since the modifications which have to be made to the theory of Paper I are relatively minor, and consist mainly in replacing the form for DT_0 appropriate to the Bénard situation [cf. Paper I, equation (53)] by (33) above, and the condition $F = 0$ on $z = 0$ of the Bénard case by the condition $DF = 0$ on $z = 0$ appropriate here.

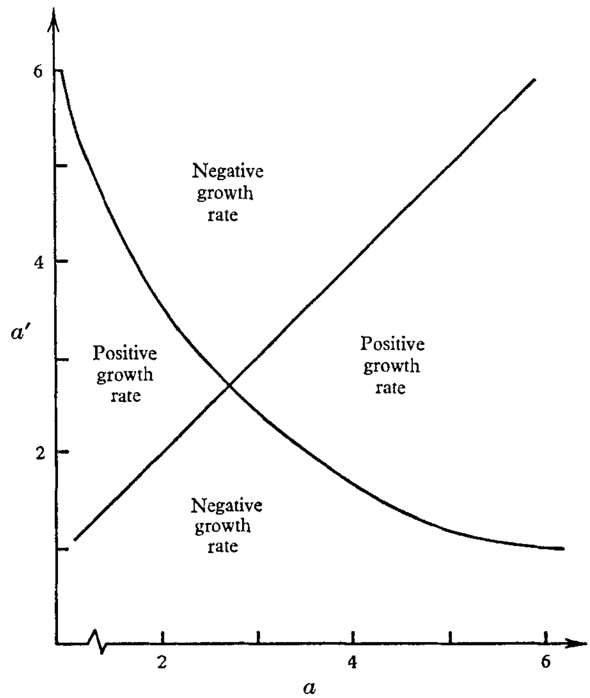

FIGURE 4. Stability diagram for convection in rolls at $R = 7000$, $P = 6{\cdot}7$. On the two curves shown, the perturbation of steady finite amplitude convection in rolls of wave-number a by a pattern of wave-number a' is marginal. Elsewhere, the steady motion is stable or unstable, as indicated. The preferred mode is located at the intersection of the curves.

We add, to a basic finite-amplitude solution (8), *perturbations* of the form $\mathbf{u}'$ and θ', where

$$\theta' = F'(z,t)f'(x,y), \quad \text{etc.} \tag{39}$$

The perturbation planform, f', has a different structure or different wave-number a' from f, or both. (If $a' \neq a$, the growth of the perturbation depends on a' but not on the structure of f'.) The normal modes of largest growth rate σ, are sought from the linear homogeneous equations governing W' and F'. If this growth rate is positive, we conclude that the steady state (W, F) selected is unstable, and could not be attained in practice. If the growth rate is negative for all choices of f', we presume it is stable. There is no *a priori* theoretical reason why there should be even one stable mode or why, if there is one, it should be unique.

First consider the perturbation of rolls by patterns of a different wave-number ($a' \neq a$). The growth rate, σ, will be a function of a and a', i.e. $\sigma = \sigma(a, a')$. It is easily shown, in fact, that $\sigma \to 0$ as $a' \to a$. In the theory given here, it is always found that σ is real. A typical situation is shown in figure 4 for $R = 7000$, $P = 6{\cdot}7$. On the two intersecting curves shown, σ is zero. (One of these is, in fact, the straight line $a = a'$.) Between these curves lie regions of stability and instability,

R	a_p	T_0	$W(\frac{1}{2})$	$F(0)$	$\sigma(R \to H)$
R_c	2·6292	0·50000	0·0000	0·00000	0·000
3,000	2·1462	0·46897	0·7815	0·02442	−0·032
4,500	2·6670	0·38399	3·5355	0·04534	−0·392
7,000	2·7144	0·31376	5·6299	0·04273	−0·965
10,000	2·7622	0·27352	7·4848	0·03723	−1·776
14,000	2·8128	0·24284	9·4862	0·03199	−10·547
18,000	2·8534	0·22415	11·1911	0·02850	−10·707
25,000	2·9102	0·20310	13·7552	0·02458	−10·967
35,000	2·9726	0·18456	16·8599	0·02123	—
45,000	3·0224	0·17226	19·5568	0·01909	—
55,000	3·0984	0·16692	22·0951	0·01740	—
60,000	3·1373	0·15914	22·805	0·01670	—
70,000	3·3205	0·15196	25·936	0·01511	—

TABLE 3. Roll patterns: preferred wave-numbers

as shown. For all values of a, except one, there exists a range of values of a' for which $\sigma > 0$, i.e. rolls of these a are unstable. For one value of a ($\approx 2{\cdot}71$) at which the curves cross, the roll is (neutrally) stable, i.e. $\sigma \leqslant 0$ for all a', with equality only if $a' = a$. A method by means of which this value, a_p, of a can be located directly, without the necessity of constructing a figure such as figure 4, is given in Paper I. When applied to the present problem it gives the results listed in table 3. It should be emphasized that, although in general σ depends on P, the location of a_p is entirely independent of Prandtl number.

Next consider the perturbations of hexagons by patterns of a different wave-number ($a' \neq a$). Again the growth rate $\sigma(a, a')$ appears to be real. In this case, however, it does not tend to zero as $a' \to a$. Two typical situations, both for down-hexagons, are shown in figure 5 for $R = 7000$, and in figure 6 for $R = 10{,}000$ ($P = 6{\cdot}7$). For the former of these, there exists, for all a, a band of wave-numbers a' for which $\sigma > 0$. All down-hexagons are therefore unstable at $R = 7000$. For $R = 10{,}000$, however, there is a range $a_1 \leqslant a \leqslant a_2$ of a in which $\sigma \leqslant 0$, i.e. for which the hexagons are completely stable. In fact, except at the end-points of this range, $\sigma < 0$ for all a'. This contrast with the neutral stability of the rolls, suggests that these patterns would have more permanence than rolls in any practical situation. As R increases, the band $[a_1, a_2]$ of stable a widens, as shown in figure 7, a figure curiously reminiscent of the neutral stability curve (figure 1). It appears from this figure, that down-hexagons are, on the present theory, stable for all R greater than 8750 (approx.). Some details concerning the down-hexagons

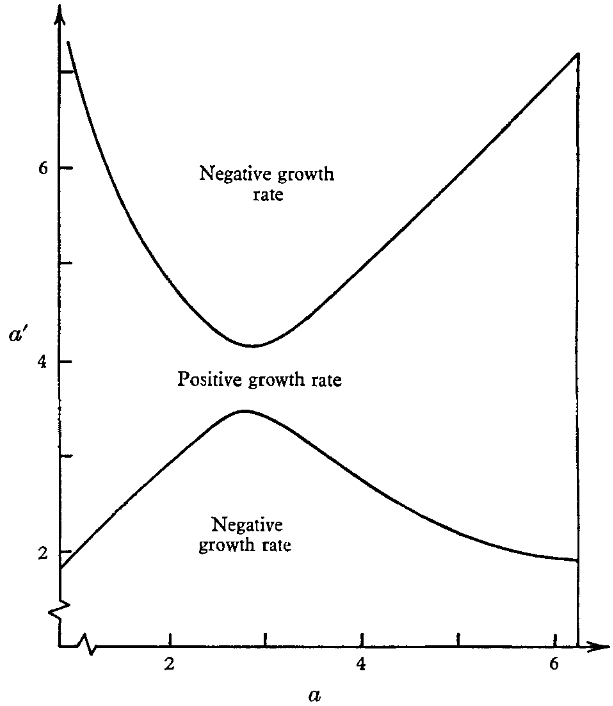

FIGURE 5. Stability diagram for convection in down-hexagons at $R = 7000$, $P = 6{\cdot}7$. On the two curves shown, the perturbation of steady finite amplitude convection in down-hexagons of wave-number a by a pattern of wave-number a' is marginal. Between the curves, the perturbation grows.

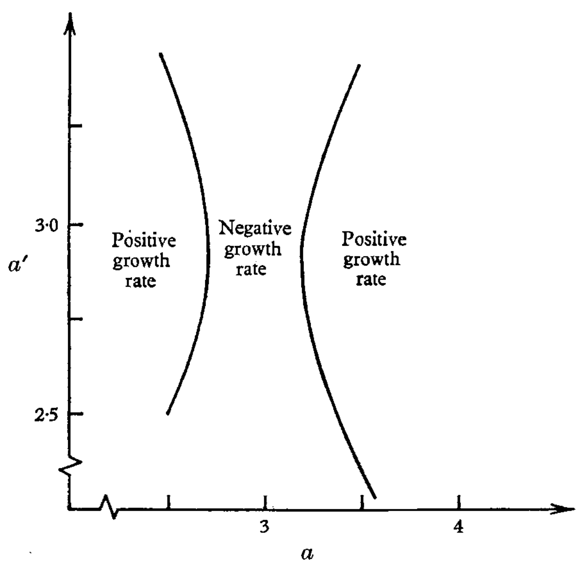

FIGURE 6. Stability diagram for convection in down-hexagons at $R = 10{,}000$, $P = 6{\cdot}7$. On the two curves shown, the perturbation of steady finite amplitude convections in down-hexagons of wave-number a by a pattern of wave-number a' is marginal. Between the curves, the perturbation decays, i.e. down-hexagons are stable for a finite band of wave-numbers a.

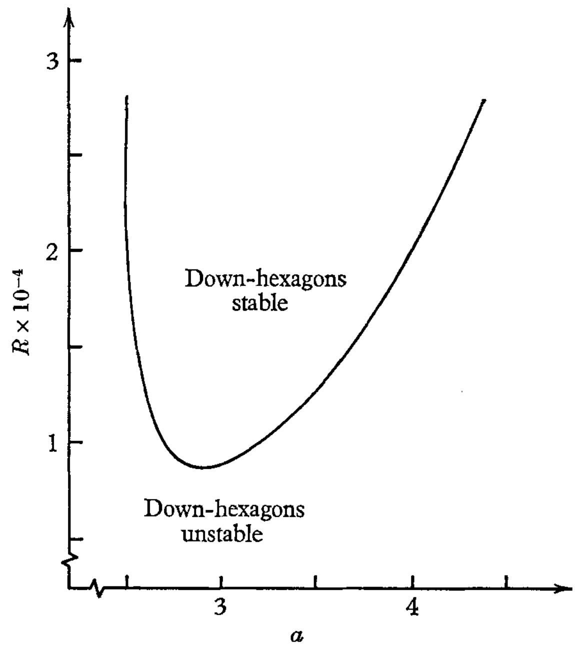

FIGURE 7. Overall stability diagram for down-hexagons. The figure, which is a composite of many figures of the type illustrated in figure 6, shows for each R the range of values of a within which stable down-hexagons exist.

R	$-T_0$	$-W(\frac{1}{2})$	$-F(0)$	$\sigma(H \to R)$	a_1	a_2
7,000	0·31493	5·9095	0·053635	−0·347	—	—
8,000	0·29743	6·6072	0·052735	−0·456	—	—
9,000	0·28332	7·2459	0·051539	−0·576	2·82	3·03
10,000	0·27165	7·8388	0·050219	−0·705	2·71	3·20
14,000	0·23957	9·8956	0·045149	−1·321	2·57	3·61
21,000	0·20926	12·7822	0·038639	−10·778	2·52	4·07
28,000	0·19189	15·1661	0·034368	−11·009	2·51	4·38

TABLE 4. Hexagonal patterns for $a = 3$: motion downwards in centre ($P = 6{\cdot}7$)

a	$-T_0$	$-W(\frac{1}{2})$	$-F(0)$	$\sigma(H \to R)$
1	0·33563	3·2516	0·125640	−0·049
2	0·24583	7·3546	0·070946	−0·656
3	0·23957	9·8956	0·045149	−1·321
4	0·25110	10·7518	0·032090	−1·342
5	0·27656	10·2304	0·023431	−1·033
6	0·32340	8·7381	0·016231	−0·668
7	0·39211	6·4156	0·009413	−0·375
8	0·48603	2·1326	0·002057	−0·140

TABLE 5. Down-hexagons ($P = 6{\cdot}7$, $R = 14{,}000$)

are listed in tables 4 and 5 including the limits $a_1(R)$ and $a_2(R)$ of a between which stable down-hexagons exist.

We have also examined the stability of up-hexagons by patterns of a different wave-number, and have, for every a and R for which we have performed the calculation, invariably found that σ is positive over a wide band of a'.

Finally, we consider the perturbation of rolls (or down-hexagons) by hexagons (or rolls), of the same wave-number ($a' = a$). In this case, there seemed to be some evidence that, as R increases, the dominant normal mode of the (W', F') perturbation becomes increasingly oscillatory in character. Indeed our numerical procedure, which could only locate normal modes of real σ, gave unexpected results between $R = 14{,}000$ and $R = 21{,}000$ for rolls ($a = a_p$), and between $R = 10{,}000$ and $R = 14{,}000$ for down-hexagons $a = 3$; see tables 3 and 4. We interpret the apparent discontinuities in $\sigma(H \rightarrow R)$ and $\sigma(R \rightarrow H)$, (the growth rates for roll perturbations superimposed on hexagonal states, and for hexagonal perturbations superimposed on steady roll convection) as follows. We suppose that, for the larger values of R, our numerical procedure has located the largest growth rate amongst the normal modes possessing real σ, probably the one with the third greatest real part (since two real roots must coalesce before the complex conjugate pair is formed); we conjecture that the modes with greatest growth rate are of complex σ, and therefore not located by our search, but this is an open question.

Summarizing our theoretical results, we have found (for water, $P = 6{\cdot}8$) that, in the range $2772 < R < 8750$ (approximately), there is only one possible convection pattern, viz. rolls of a certain definite wave-number, a_p (see table 3); these are marginally stable.† For $R > 8750$, there exists, in addition, a range (a_1, a_2) of values of a, whose bounds a_1 and a_2 depend on R (and P) in which motion in down-hexagons is possible; since these are completely stable (as distinct from the marginal stability of the roll solution), we conclude that these would be more likely to be observed in practice. There is no method, on the present theory, of deciding which (if any) of the values of a in (a_1, a_2) is preferred.

Tritton & Zarraga have made no effort to establish the critical value 2772 for marginal convection, nor have they studied in detail convective motions in the range $R < 12{,}000$; the one observation they report (for $R \approx 7000$) is visual, and does not confirm the theoretical expectation of convection in rolls. It may be observed, however, that for $R = 12{,}000$, the smallest for which they present photographs, there is some evidence for a roll-like structure in addition to the more obvious hexagonal pattern. A more serious discrepancy lies in the fact that the wave-numbers of the hexagonal patterns they observe are definitely to the left ($a < a_1$) of the theoretical curve shown in figure 7: they are, then, unstable on our theory. Moreover, even supposing that a refinement of the present theory

† I am grateful to Dr F. Busse for pointing out that, according to his expansion method, the preferred mode of convection near critical should be hexagonal; in fact, if R. Krishnamurti's (1967) work in a similar situation is a guide, they will be up-hexagonal. If this is the case, there will exist a band of R in which rolls only are stable (Busse 1967), as, indeed, we have concluded here. The main difference will be that this band will not extend from 8750 (approx) down to R_c, but only down to an R a little above R_c.

would disclose that a_1 should be smaller, it would be difficult to account for the preference for smaller wave-numbers of the band (a_1, a_2).

This research was sponsored by the Science Research Council under Grant B/SR/150. My primary thanks are due to Dr K. Wright for undertaking the far from straightforward task of solving (41), cf. figures 2 and 3 above. I am grateful to Miss M. F. Tabrett for her conscientious assistance in the project, and to Dr S. A. Piacsek for help in obtaining tables 1 and 2. Dr Tritton kindly communicated the experimental results ahead of their publication. My thanks are due to Professor E. S. Page, director of the Computing Laboratory of my University, for generous support. I have also to thank Professor J. T. Stuart, Dr F. Busse and Dr D. C. Tozer for some discussions.

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